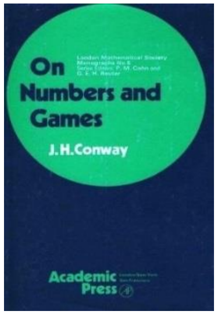
Scoring Mean Value Theorem

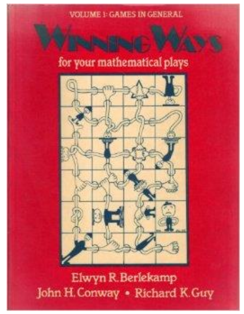
Carlos Pereira dos Santos
Richard Nowakowski
Urban Larsson

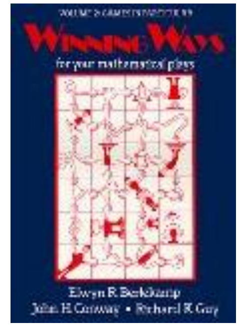
Combinatorial Game Theory Colloquium II

1. Standard Combinatorial Game Theory

Normal-Play convention


On
Numbers and
Games
J.H.Conway
Academic
Press


VOLUME 1: GAMES IN GENERAL
Winning Ways
for your mathematical plays
Elwyn R. Berlekamp
John H. Conway • Richard K. Guy


Winning Ways
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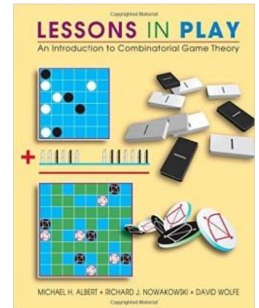
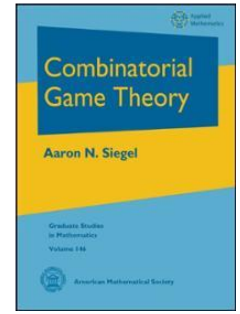

LESSONS IN PLAY
An Introduction to Combinatorial Game Theory
MICHAEL H. ALBERT • RICHARD J. NOWAKOWSKI • DAVID WOLFE


Combinatorial
Game Theory
Aaron N. Siegel
Graduate Studies
in Mathematics
American Mathematical Society

$\varnothing$

∅ «Before creation»

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Day 0 $\{\varnothing \mid \varnothing\}$ (by definition, zero)

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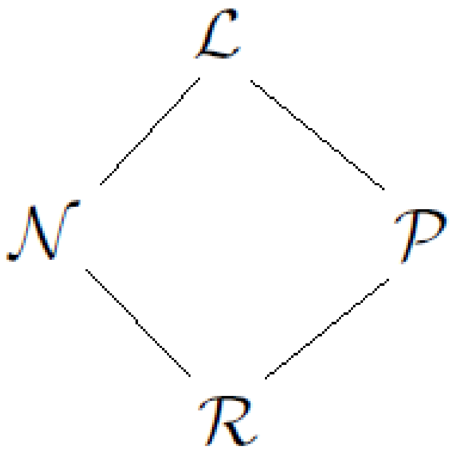
(...)

Outcomes



Order (**subordinate**)

G=H iff o(G+X)=o(H+X), for all X

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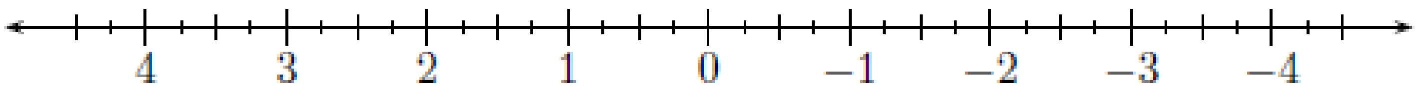
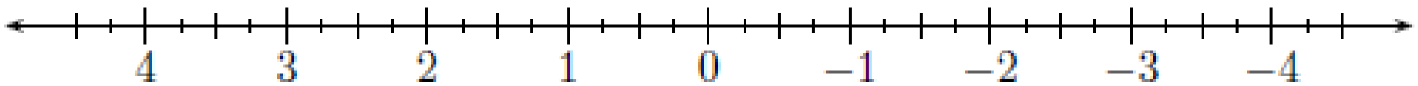
G≥H iff o(G+X) ≥ o(H+X), for all X

We have **numbers** and we have **games**

Game line: how it works?


4
3
2
1
0
−1
−2
−3
−4

There are games such that **μ(Confusion Interval)=0**



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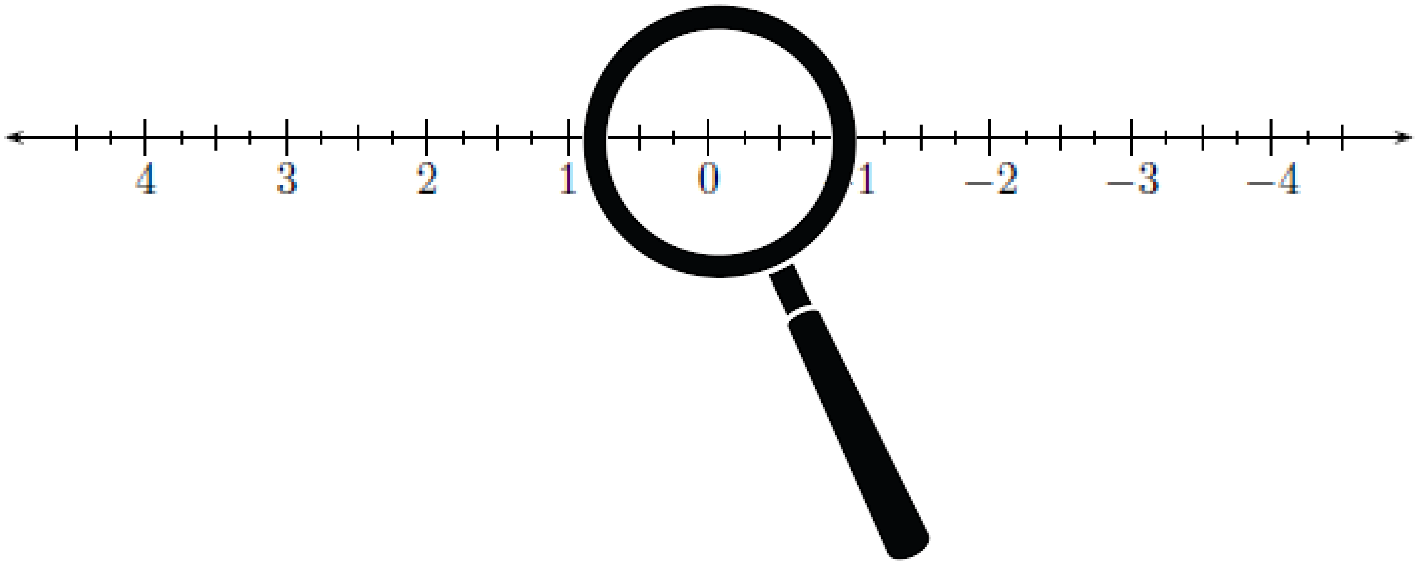

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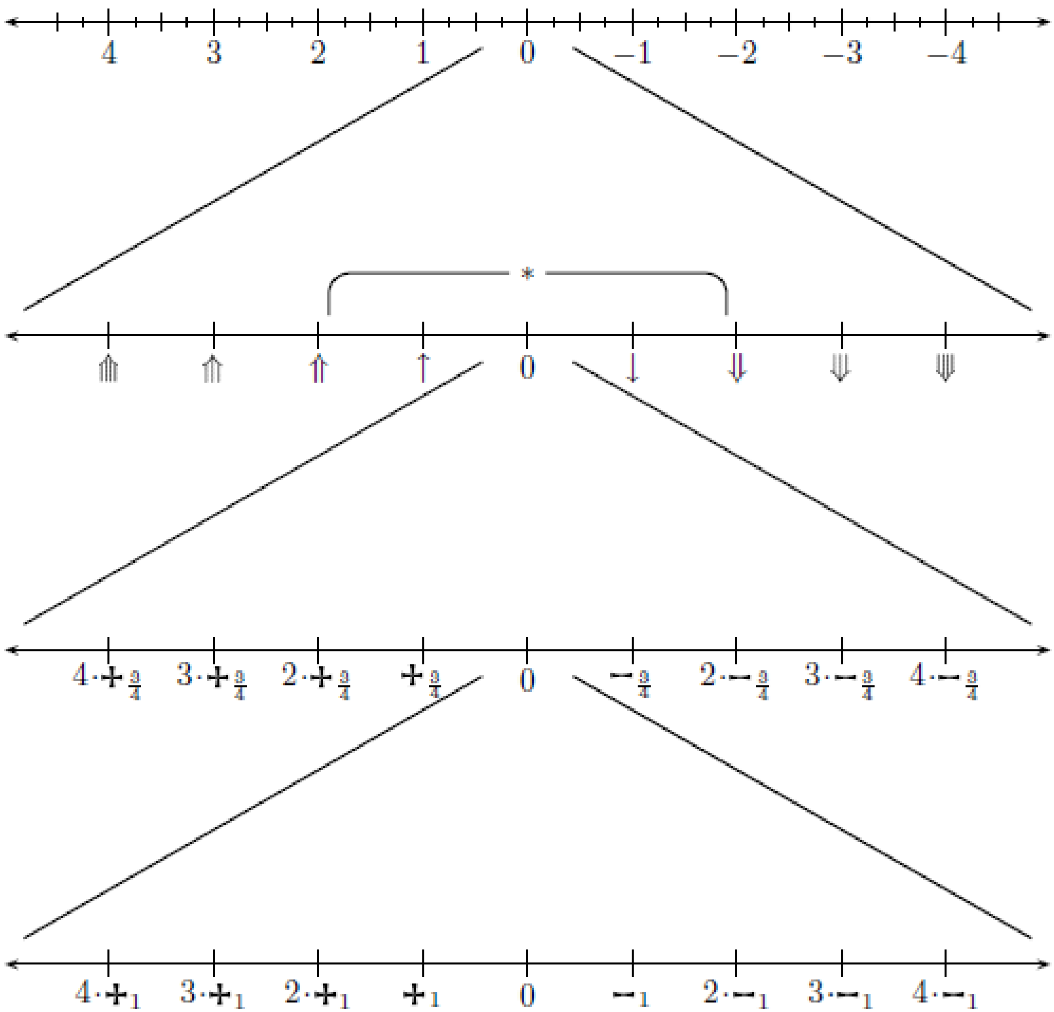

There are games such that **μ(Confusion Interval)>0**

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We can win moves there!

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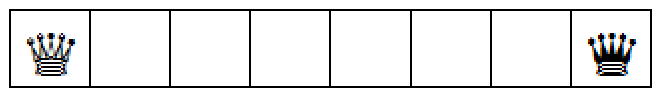

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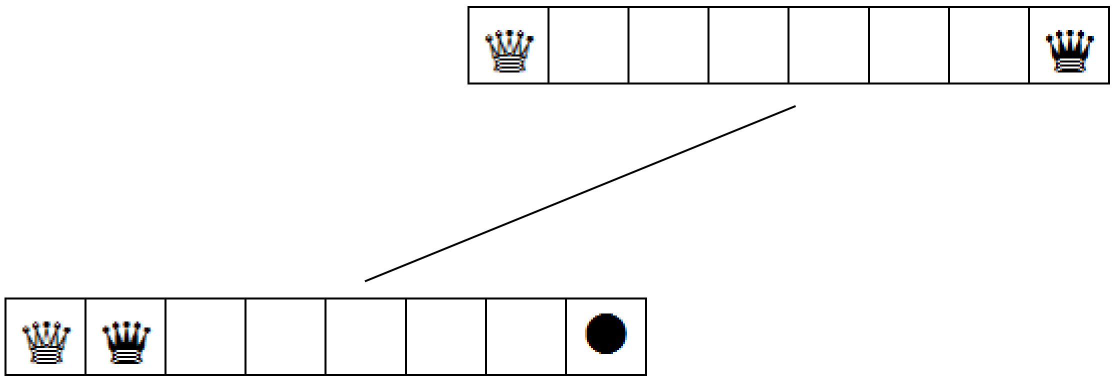

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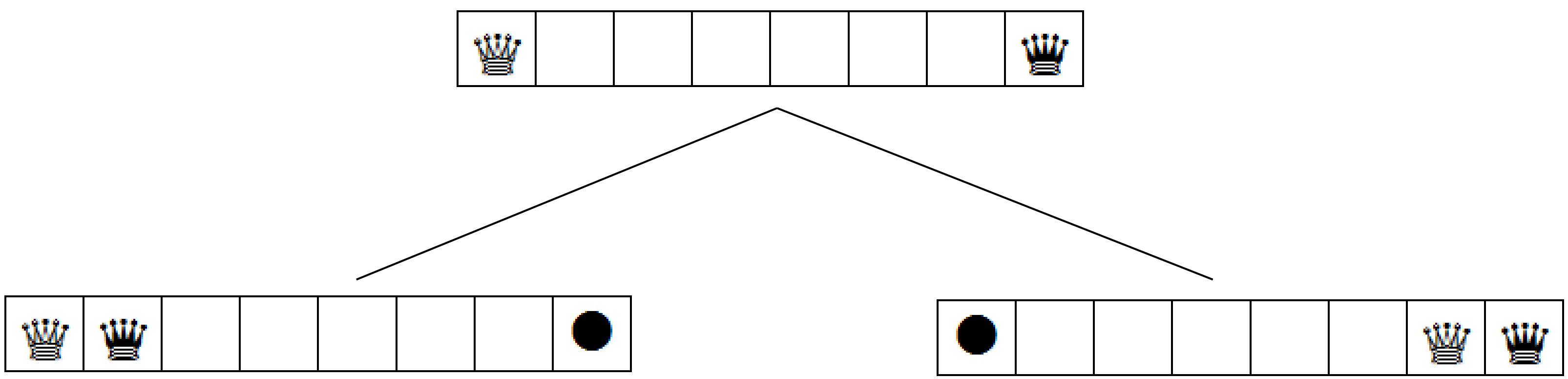

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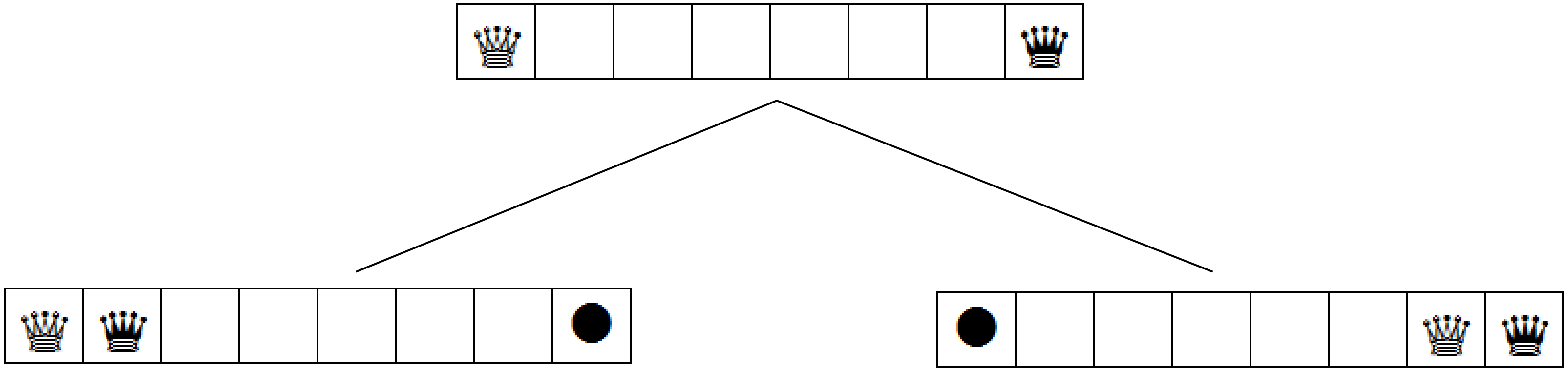
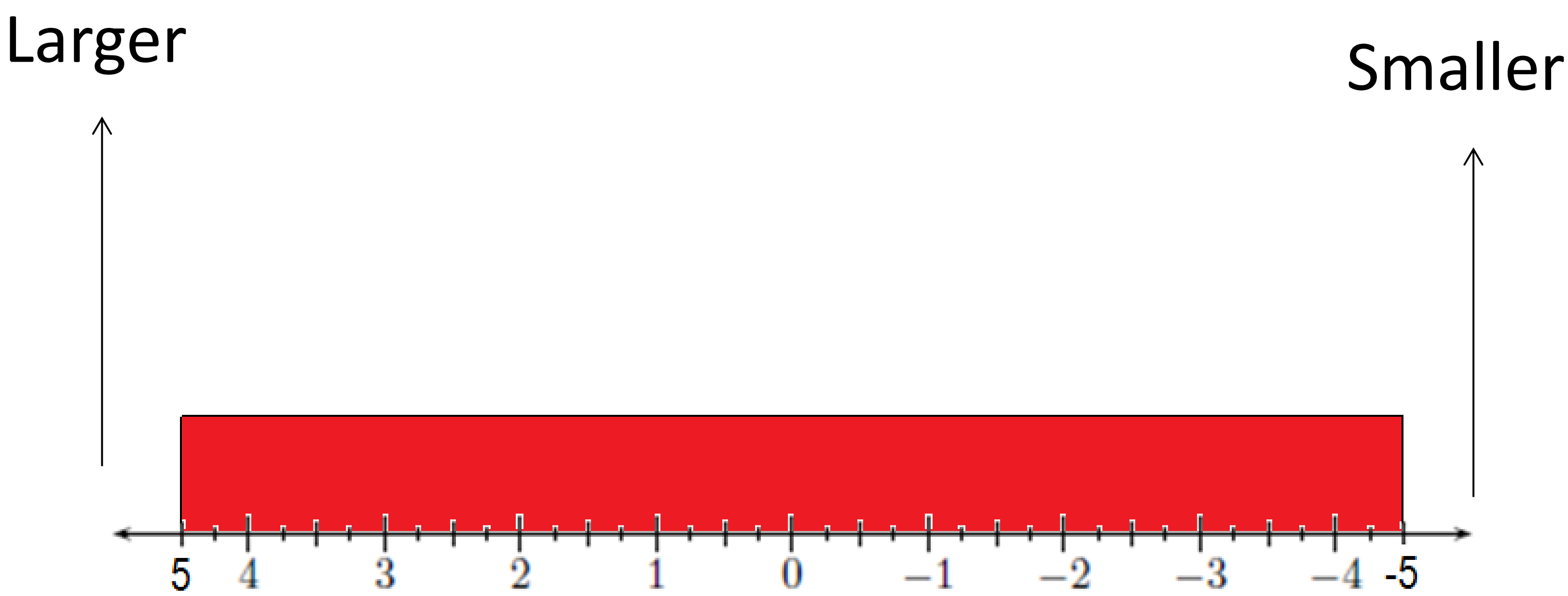


A typical hot game is {5 | -5}

In game line?



Larger
Smaller
5
4
3
2
1
0
−1
−2
−3
−4
-5

Two natural questions:

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Have we a «fair value» for a hot game?
That is, is there a **number** g such that

n.g – perturbation < n.G < n.g + perturbation ?

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Yes! Mean value theorem

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Yes! More or less…

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Yes! More or less…

Temperature theory (thermography)

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Mean value theorem

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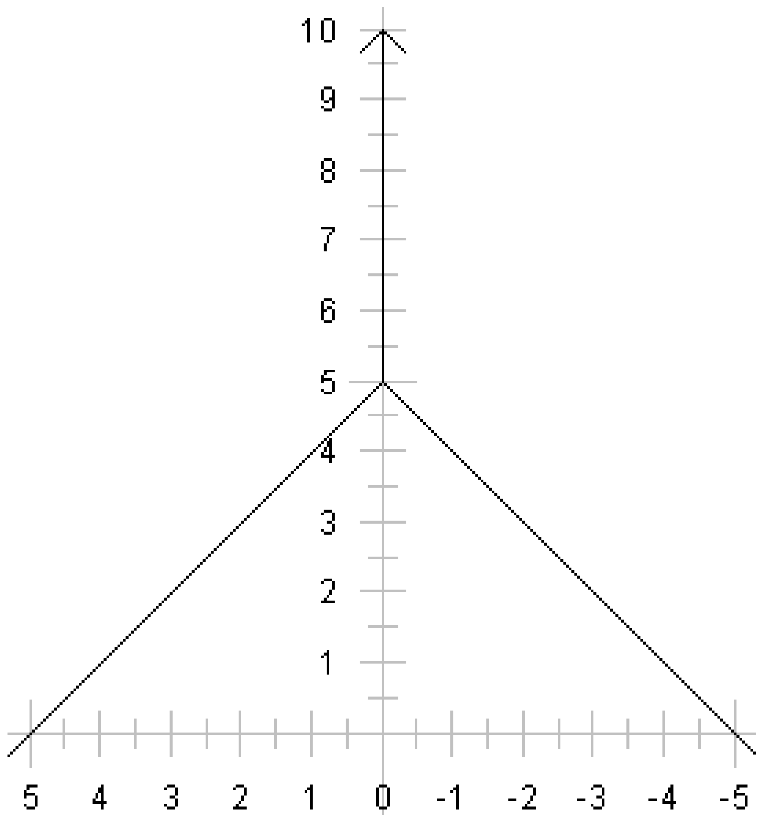
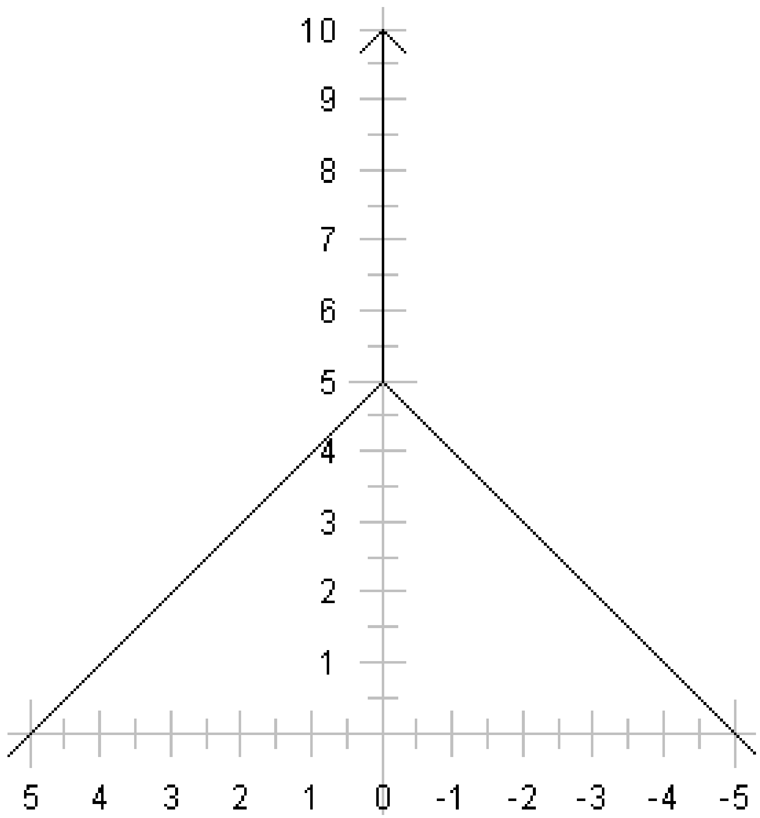
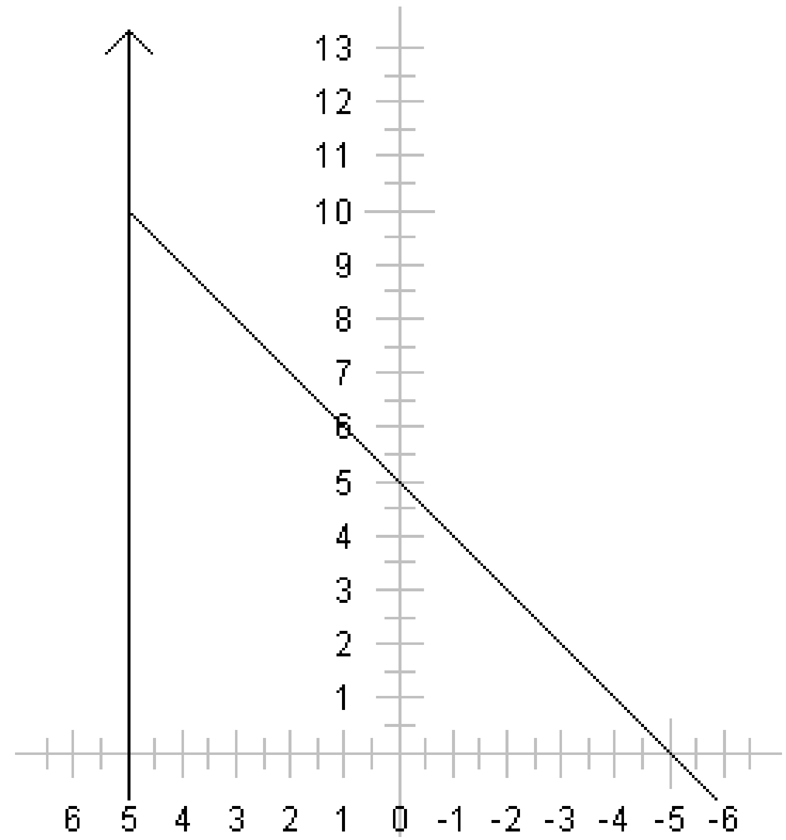
Temperature theory (thermography)



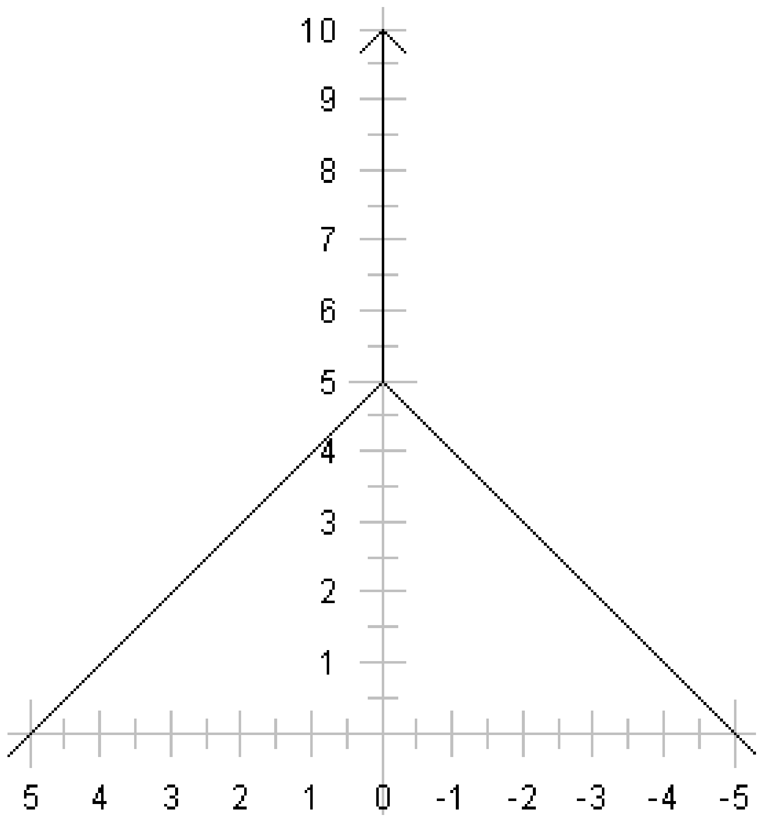
G={5|−5}


10
9
8
7
6
5
4
3
2
1
5 4 3 2 1 0 -1 -2 -3 -4 -5

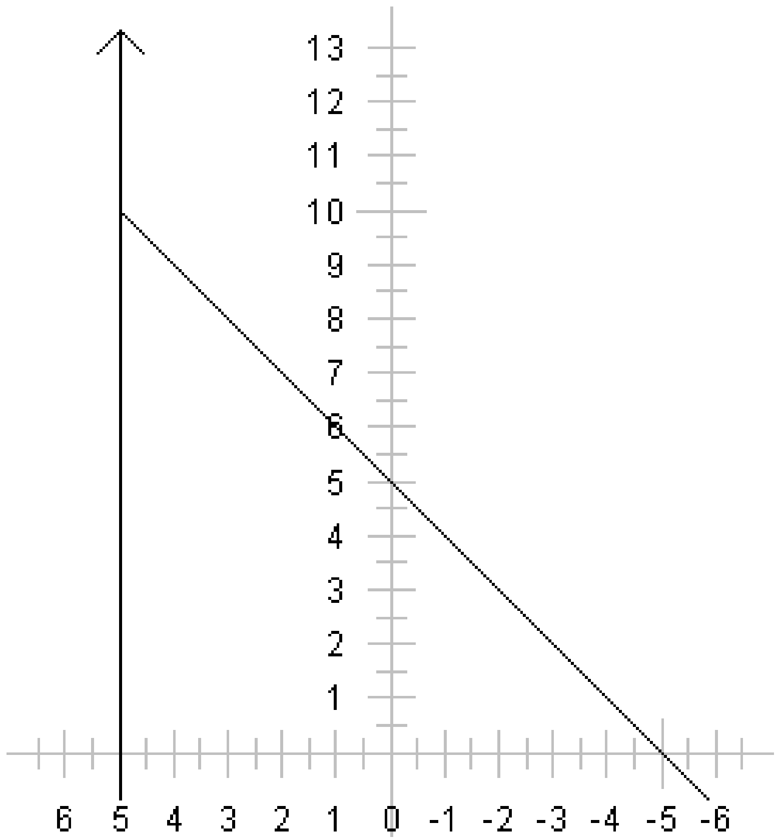
G={5|-5}


13
12
11
10
9
8
7
6
5
4
3
2
1
6 5 4 3 2 1 0 -1 -2 -3 -4 -5 -6

H={{1000|5}|-5}



G={5|-5}



H={{1000|5}|-5}

Sente

2. Scoring Combinatorial Game Theory

$\varnothing^a$ «Before creation»

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Day 0 $<\varnothing^0 \mid \varnothing^0>$ (0)

$\varnothing^a$ «Before creation»

Day 0 $<\varnothing^0 \mid \varnothing^0>$ (0) $<\varnothing^1 \mid \varnothing^1>$ (1)

$\varnothing^{a}$ «Before creation»

Day 0 $<\varnothing^{0} \mid \varnothing^{0}>$ (0) $<\varnothing^{1} \mid \varnothing^{1}>$ (1) $<\varnothing^{-2} \mid \varnothing^{4}>$ (...)

$\varnothing^a$ «Before creation»

Day 0 $\langle \varnothing^0 \mid \varnothing^0 \rangle$ (0) $\langle \varnothing^1 \mid \varnothing^1 \rangle$ (1) $\langle \varnothing^{-2} \mid \varnothing^4 \rangle$ (...)

Day 1 $\langle 0 \mid \varnothing^0 \rangle$ $(\overline{1})$ $\langle 1 \mid -1 \rangle$ $\langle 1 \mid \varnothing^1 \rangle$ (...)

$\varnothing^{a}$ «Before creation»

Day 0 $<\varnothing^{0} \mid \varnothing^{0}>$ (0) $<\varnothing^{1} \mid \varnothing^{1}>$ (1) $<\varnothing^{-2} \mid \varnothing^{4}>$ (…)

Day 1 $<0 \mid \varnothing^{0}>$ $(\bar{1})$ $<1 \mid -1>$ $<1 \mid \varnothing^{1}>$ (…)

Day 2 $<\bar{1} \mid -\bar{1}>$ (…)

$\varnothing^a$ «Before creation»

Day 0 $<\varnothing^0 \mid \varnothing^0>$ (0) $<\varnothing^1 \mid \varnothing^1>$ (1) $<\varnothing^{-2} \mid \varnothing^4>$ (...)

Day 1 $<0 \mid \varnothing^0>$ ($\overline{1}$) $<1 \mid -1>$ $<1 \mid \varnothing^1>$ (...)

Day 2 $<\overline{1} \mid -\overline{1}>$ (...)

Guaranteed Property

Outcomes

Scores

In Scoring CGT, we have a **«symbiotic» nature points/moves**

Consider G=< <5|5>, -20 || 0>

Consider G=< <5|5>, -20 || 0>

To win points, of course, <5|5> is the best Left option.

Consider G=< <5|5>, -20 || 0>

But, imagine the situation G + <-100|100>…

Consider G=< <5|5>, -20 || 0>

But, imagine the situation G + <-100|100>

The good Left option is -20+ <-100|100> and…

Consider G=< <5|5>, -20 || 0>

But, imagine the situation G + <-100|100>

The good Left option is -20+ <-100|100> and

Right must «open» the *zugzwang*

LS<RS (*zugzwang*)

A game is a *zugzwang* if $LS(G)<RS(G)$

In SCGT, *zugzwangs* are important!

In Scoring CGT, the **analogous concept of numbers** is

$$p + \bar{m}$$

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$$p + \overline{m}$$

A game like

$$3 - \overline{2}$$

is stabilized both in terms of points and moves

Some examples

A game like

$$\langle 3 + \bar{2} \mid 3 - \bar{2} \rangle$$

is cold in terms of points and hot in terms of moves

A game like

$$\langle 3 + \overline{2} \mid 3 - \overline{2} \rangle$$

is cold in terms of points and hot in terms of moves

A game like

$$\langle 3 \mid -2 \rangle$$

is cold in terms of moves and hot in terms of points

Stops are different if a player **can wait or not**

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Consider

G=< <<-3|3>|<-2|2>> | 1>

Stops are different if a player **can wait or not**

Consider

$$G=< <<-3|3>|<-2|2>> \quad | 1>$$

If players cannot wait, LS(G)=-2

Stops are different if a player **can wait or not**

Consider

$$G=< <<-3|3>|<-2|2>> \quad | 1>$$

If players cannot wait, $LS(G)=-2$

If Left can wait, $\overline{LS}(G)=2$

Stops are different if a player **can wait or not**

Consider

$$G=\langle \langle\langle -3|3\rangle|\langle -2|2\rangle\rangle \mid 1\rangle$$

If players cannot wait, $LS(G)=-2$

If Left can wait, $\overline{LS}(G)=2$

If Right can wait, $\underline{LS}(G)=-3$

What we know so far about Scoring CGT?

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Conway's embedding

What we know so far about Scoring CGT?

Conway’s embedding
Reductions

What we know so far about Scoring CGT?

Conway’s embedding
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Canonical forms

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Structure (conjugate property)

What we know so far about Scoring CGT?

Conway’s embedding
Reductions
Canonical forms
Structure (conjugate property)
Unsubordinated comparison

3. What about confusion zones and mean values in SCGT?

Bounds

Bounds

Consider

$G=\langle \quad \langle 10 \mid 5, 4+\overline{3} \rangle \quad \| \quad \langle -3-\overline{1*}, -2-\overline{2} \mid \overline{-1+*} \rangle \quad \rangle$

Bounds

$$G=\langle \quad \langle 10|5,\ 4+\overline{3}\rangle \quad || \quad \langle -3-\overline{1*},\ -2-\overline{2}\ |\ \overline{-1+*}\rangle \quad \rangle$$

On one hand, bounding **with scores**,

Bounds

$$G=\langle \; \langle 10 \mid 5,\ 4+\bar{3} \rangle \;\;\|\;\; \langle -3-\overline{1*},\ -2-\bar{2} \mid \overline{-1+*} \rangle \; \rangle$$

$$G \leq \langle \langle 10 \mid 10,\ 10+\bar{3} \rangle \;\mid\; \langle 10-\overline{1*},\ ,\ 10-\bar{2} \mid \overline{-1+*} \rangle \rangle$$

Bounds

$$G=\langle \quad \langle 10|5,\ 4+\overline{3}\rangle \quad || \quad \langle -3-\overline{1*},\ -2-\overline{2}\ |\ \overline{-1+*}\rangle \quad \rangle$$

Bounding **with scores**,

$$-3+\overline{\{*|-1\}} \leq G \leq 10+\overline{\{*|-1\}}$$

Bounds

$G = \langle \quad \langle 10 \mid 5,\ 4+\overline{3} \rangle \quad \| \quad \langle -3-\overline{1*},\ -2-\overline{2} \mid \overline{-1+*} \rangle \quad \rangle$

On the other hand, bounding **with moves**,

Bounds

$$G=\langle \quad \langle 10 \mid 5,\ 4+\overline{3} \rangle \quad \| \quad \langle -3-\overline{1*},\ -2-\overline{2} \mid \overline{-1+*} \rangle \quad \rangle$$

Maximum number of points that Left can win, even if she can wait.

Bounds

$$G=\langle \quad \langle 10 | 5,\ 4+\bar{3} \rangle \quad || \quad \langle -3-\overline{1*},\ -2-\bar{2} \mid \overline{-1+*} \rangle \quad \rangle$$

But she **also wins 3 moves**

Bounds

$$G = < \quad <10 \mid 5,\ 4+\overline{3}> \quad \mid\mid \quad < -3-\overline{1*},\ -2-\overline{2} \mid \overline{-1+*}> \quad >$$

Bounding **with moves**,

$$-2 -\overline{2} \leq G \leq 4 + \overline{3}$$

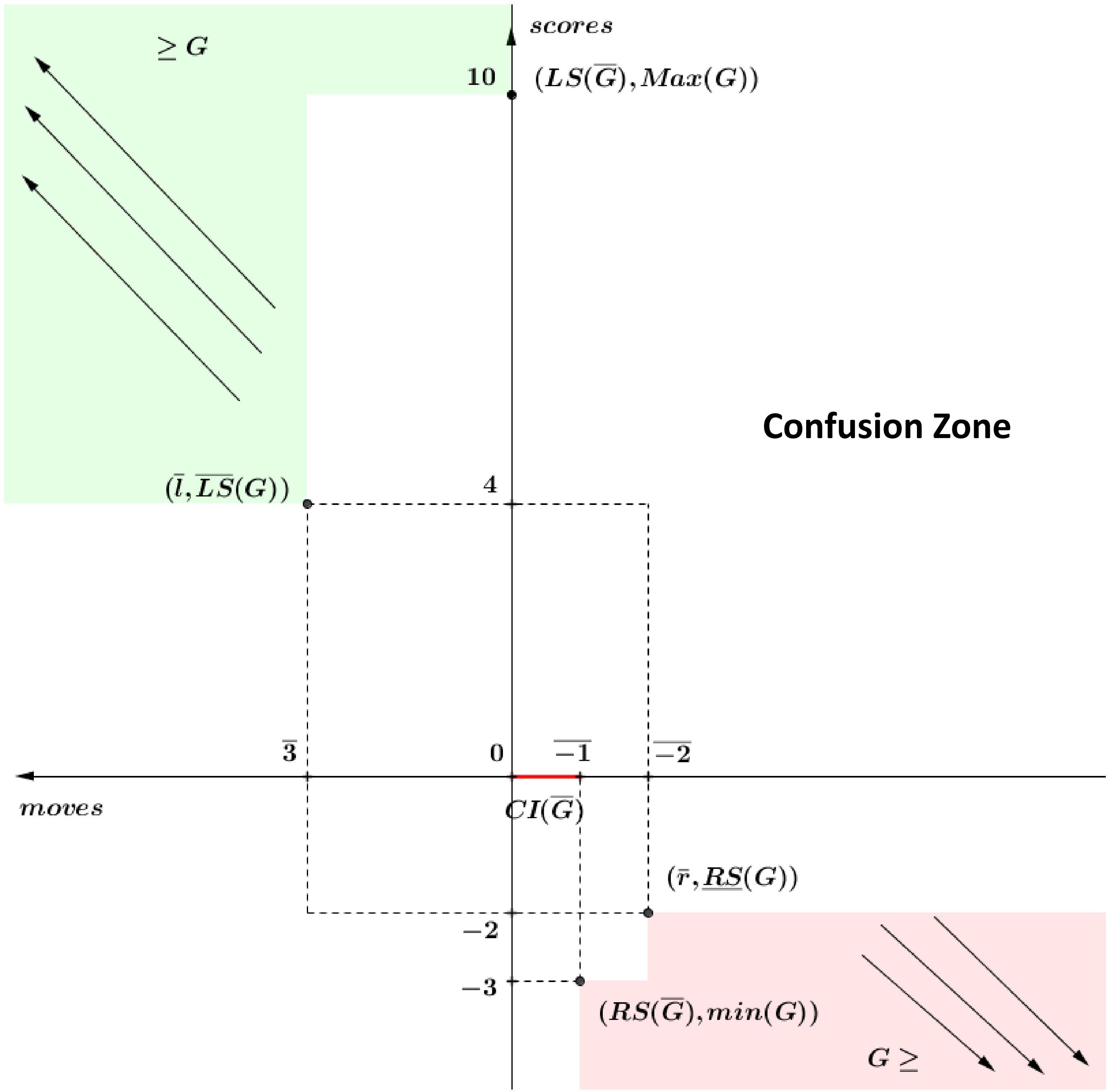
Bounds

with scores

$$\min(G) + \overline{G} \leq G \leq \mathrm{Max}(G)+\overline{G}$$

with moves

$$\underline{RS}(G) + \overline{r} \leq G \leq \overline{LS}(G) + \overline{l}$$



scores
≥ G
10
(LS(G̅), Max(G))
Confusion Zone
(l̄, L̅S̅(G))
4
3̅
0
−1̅
−2̅
moves
CI(G̅)
(r̄, RS(G))
−2
−3
(RS(G̅), min(G))
G ≥

«The question»

«The question»

Have we a «fair value» for a hot scoring game?
That is, is there a **game** $p+\overline{m}$ such that

$$\mathbf{n.(p+\overline{m}) - perturbation < n.G < n.(p+\overline{m}) + perturbation}\ ?$$

$$Min(G) \leqslant \underline{RS}(G) \leqslant RS(G) \leqslant \overline{RS}(G) \bullet \underline{LS}(G) \leqslant LS(G) \leqslant \overline{LS}(G) \leqslant Max(G)$$

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Definition G is a *zugzwang* if $LS(G) < RS(G)$.

G is a *weak zugzwang* if $\underline{LS}(G) < \overline{RS}(G)$.

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If G is a zugzwang, it is also a weak zugzwang.

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The opposite is not true.

$$Min(G) \leqslant \underline{RS}(G) \leqslant RS(G) \leqslant \overline{RS}(G) \quad \underline{LS}(G) \leqslant LS(G) \leqslant \overline{LS}(G) \leqslant Max(G)$$

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If G is a zugzwang, it is also a weak zugzwang.

The opposite is not true.

$$G = << -4|4 > \mid < -4|4 >>$$

$$Min(G) \leqslant \underline{RS}(G) \leqslant RS(G) \leqslant \overline{RS}(G) \;\bullet\; \underline{LS}(G) \leqslant LS(G) \leqslant \overline{LS}(G) \leqslant Max(G)$$

Definition G is a *zugzwang* if $LS(G) < RS(G)$.

G is a *weak zugzwang* if $\underline{LS}(G) < \overline{RS}(G)$.

If a game is not a weak zugzwang, the chain does not break

Now…

Now…

$n.(-p-\overline{m})$ – perturbation < $n.G$ < $n.(p+\overline{m})$ + perturbation

it is not a result…

Now…

$$n.(-p-\overline{m}) - \text{perturbation} < n.G < n.(p+\overline{m}) + \text{perturbation}$$

it is not a result…

$$n.(-p+\overline{m}) - \text{perturbation} < n.G < n.(p+\overline{m}) + \text{perturbation}$$

moves: it is true, comes from standard CGT

Now…

$n.(-p-\overline{m}) - \text{perturbation} < n.G < n.(p+\overline{m}) + \text{perturbation}$

it is not a result…

$n.(-p+m) - \text{perturbation} < n.G < n.(p+\overline{m}) + \text{perturbation}$

moves: it is true, comes from standard CGT

$n.(p-\overline{m}) - \text{perturbation} < n.G < n.(p+\overline{m}) + \text{perturbation}$

scores?

Now…

$n.(-p-\overline{m})$ – perturbation < n.G < $n.(p+\overline{m})$ + perturbation

it is not a result…

$n.(-p+m)$ – perturbation < n.G < $n.(p+\overline{m})$ + perturbation

moves: it is true, comes from standard CGT

$n.(p-\overline{m})$ – perturbation < n.G < $n.(p+\overline{m})$ + perturbation

scores?

$n.(p+\overline{m})$ – perturbation < n.G < $n.(p+\overline{m})$ + perturbation

both scores and moves?

Our answers so far

$$\mathbf{n.(p-\overline{m}) - perturbation < n.G < n.(p+\overline{m}) + perturbation}$$

Our answers so far

$n.(p-\overline{m}) - \text{perturbation} < n.G < n.(p+\overline{m}) + \text{perturbation}$

True iff G is not a weak zugzwang

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$$n.(p-\overline{m}) - \text{perturbation} < n.G < n.(p+\overline{m}) + \text{perturbation}$$

True iff G is not a weak zugzwang

$$n.(p+\overline{m}) - \text{perturbation} < n.G < n.(p+\overline{m}) + \text{perturbation}$$

Our answers so far

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True iff G is not a weak zugzwang

$$n.(p+\overline{m}) - perturbation < n.G < n.(p+\overline{m}) + perturbation$$

Problems…

Our answers so far

$$n.(p-\overline{m}) - \text{perturbation} < n.G < n.(p+\overline{m}) + \text{perturbation}$$

True iff G is not a weak zugzwang

$$n.(p+\overline{m}) - \text{perturbation} < n.G < n.(p+\overline{m}) + \text{perturbation}$$

Problems…

$$<\ <2+\overline{1}\,|\,\overline{1}>\ ||\ <-\overline{1}\,|\,-1-\overline{2}>\ >$$

Scoring Mean Value Theorem

Carlos Pereira dos Santos
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