Solving illuNIMati through boomerang games

Gabriel Renault

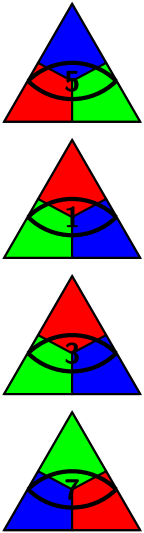
illuNIMati

A position consists in several heaps of triangles coloured blue, red and green.

A Left move is to choose a heap with blue or green facing North, remove any positive number of tokens from that heap (and possibly change the orientation).

A Right move is to choose a heap with red or green facing North, remove any positive number of tokens from that heap (and possibly change the orientation).

We call B_i (G_i, R_i) a heap of i tokens with blue (green, red) facing North.



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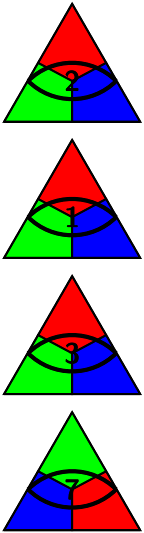

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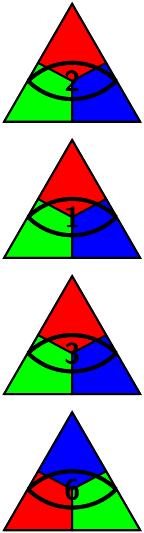

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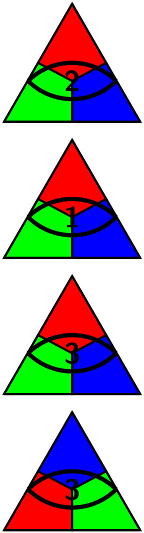

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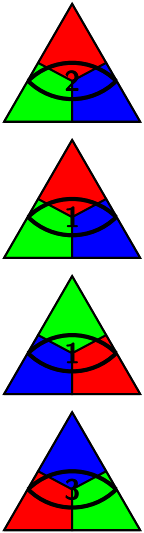

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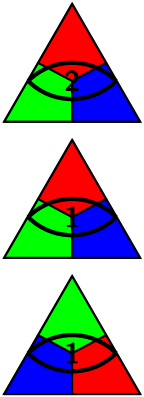

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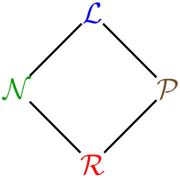
Comparison and equivalence modulo sets of games

$G \geqslant^-_{\mathcal{U}} H$: Left always "prefers" G over H when adding it to a game in $\mathcal{U}$.

Definition (Plambeck, Siegel):

$$(G \geqslant^-_{\mathcal{U}} H) \Leftrightarrow (\forall X \in \mathcal{U}, o^-(G+X) \geqslant o^-(H+X))$$

If $((G \geqslant^-_{\mathcal{U}} H)$ and $(H \geqslant^-_{\mathcal{U}} G))$, we say $(G \equiv^-_{\mathcal{U}} H)$.



Boomerang games

Definition:

A Left *br*-end G is a Left end such that any Right option of G is either a Left end or has a Left option to a Left end.

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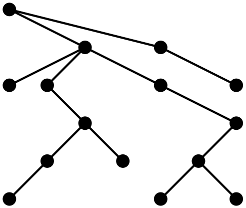
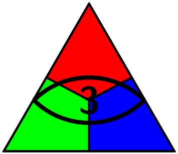
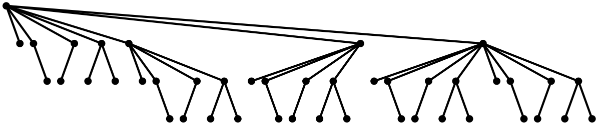


Boomerang games: examples





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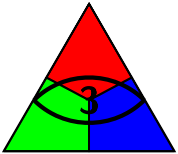
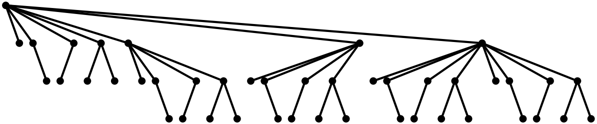
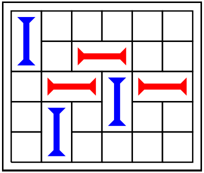
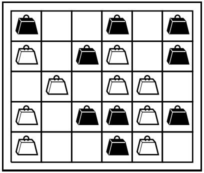
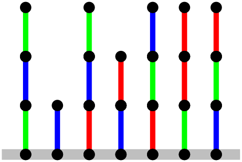


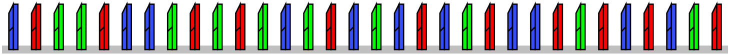


Every dead-ending game or alternating game is a boomerang game.









Some end properties of boomerang games

Lemma:

Let G be a boomerang Left end. $o^-(G + \overline{1}) = \mathcal{L}$.

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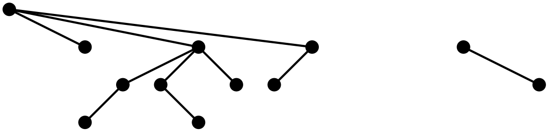

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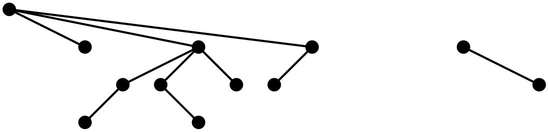

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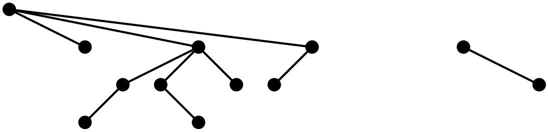

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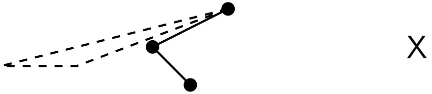
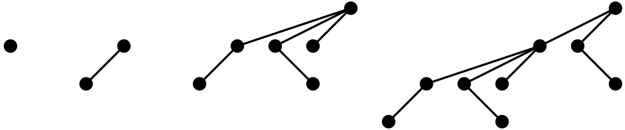


Extremal boomerang ends

Definition:

We recursively define a sequence of games as follows: $E_0 = 0$, $E_1 = 1$, $E_2 = \{0, 1, \overline{1} | \cdot\}$ and $\forall n \geqslant 3, E_n = \{E_{n-1}, \overline{1} | \cdot\}$.



E_0 E_1 E_2 E_3

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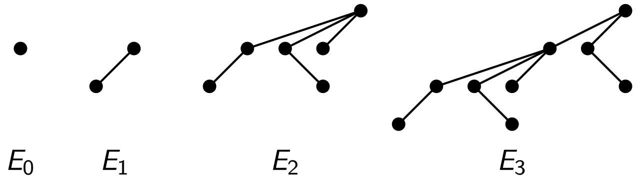


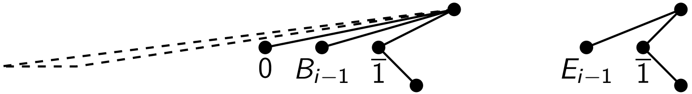
Property:

Let G be a boomerang Right end and $n \geqslant 2$, $birth(G)$ an integer.
$G \leqslant^-_{\widehat{\mathcal{B}}} E_n$.

Simplifying illuNIMati

Lemma:

$\forall i, B_i \equiv^{-}_{\widehat{\mathcal{B}}} E_i$.



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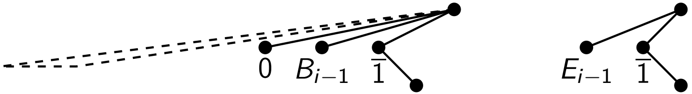

Lemma:

$\forall i \geqslant 3, G_i \equiv^{-}_{\widehat{\mathcal{B}}} \{B_{i-1}, \overline{1}, * | R_{i-1}, 1, *\}.$

Auxiliary functions

We write $G = \sum_{i \in \mathbb{N}} b_i B_i + g_i G_i + r_i R_i$.

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$f(G) = \sum_{i\geqslant 2}(b_i \cdot (i-1))$

$g(G) = n - 2 - g(G - G_n)\ (n = max\{i | g_i \neq 0, i \geqslant 3\})$

$p_1(G) = (b_1 + g_1 + r_1)[2]$

$p_2(G) = g_2[2]$

$p_3(G) = (\sum_{i\geqslant 3} g_i)[2]$

Solving illuNIMati

Theorem:

Let G be an illuNIMati position. Left wins G playing first iff:

$$\begin{cases} r_1 \geqslant b_1 + g_1 \\ b_1 < r_1 + g_1; \begin{cases} f(G) + g(G) > f(\overline{G}) \\ f(G) + g(G) = f(\overline{G}); \begin{cases} p_2(G) = 1; p_3(G) = 0 \\ p_2(G) = 0; p_1(G) + p_3(G) \neq 1 \end{cases} \end{cases} \\ b_1 = r_1 + g_1; \begin{cases} f(G) > f(\overline{G}) + g(G) \\ f(G) = f(\overline{G}) + g(G); \begin{cases} p_2(G) = 1; p_3(G) = 1 \\ p_2(G) = 0; p_1(G) + p_3(G) \neq 1 \end{cases} \end{cases} \end{cases}$$

Perspectives

- Study other games modulo boomerang games
- Look at other properties of boomerang games
- Find bigger sets of games where equivalence is larger than in the general universe

Questions?

Thank you.