The OSLO Game of Hives

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Rules

HIVES is played on a graph G with an independent set of vertices S. We think of the vertices in the set S as being coloured red.

- On Right's turn, they may colour an uncoloured vertex.
- On Left's turn, they select two adjacent vertices and swap their colours.
- Game ends when the coloured vertices form a *maximal* independent set.

If Left chooses two uncoloured vertices, their move is a pass. Therefore, the game is One-Sided LOopy (OSLO).

Sample of Play

The starting graph.

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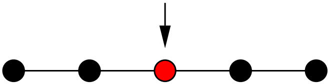
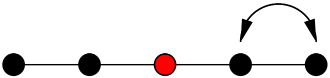


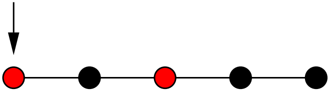
Right plays first.

Sample of Play



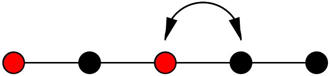
Left's move is equivalent to a pass.

Sample of Play



Right must play on either end.

Sample of Play



Left swaps this pair...

Sample of Play

Left swaps this pair... and wins!

Properties

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Theorem

- $G_n = 0$
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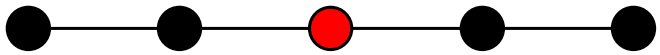
Proof.

$$G_{n-1} = \{G_{n-1} | G_n\} = \{pass | 0\} = \text{under}$$
$$G_k = \{G_k | G_k + 1\} = \{pass | \text{under}\} = \text{under}$$

□

Values



$$= \{pass|A\} = A + A$$

$$= \{A + A|0\} = A = \uparrow^{[ON]} *$$

$$= \{A|0\} = \downarrow_{[ON]}$$

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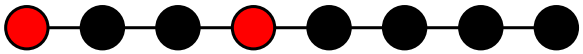

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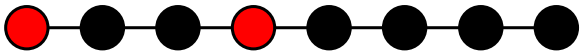

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We can construct positions of value $k \cdot A$ and $k \cdot B$ on a single path for any k.

Outcome classes

- $k \cdot A \in \mathcal{L}$ if $k \geq 2$
- $A \in \mathcal{N}$
- $\downarrow_{[ON]} \in \mathcal{R}$
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Lemma

$$i \cdot A + j \cdot B + k \cdot \downarrow_{[ON]} = \begin{cases} (i-j) \cdot A & \text{if } i > j \\ (j-i) \cdot B & \text{if } i < j \\ \downarrow_{[ON]} & \text{if } i = j \end{cases}$$

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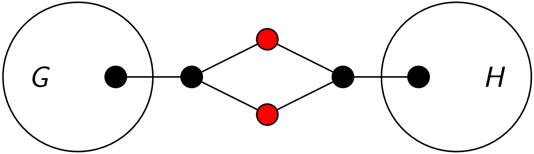
So, A acts like 1, B acts like -1 and $\downarrow_{[ON]}$ acts like 0.
Unfortunately, $\{pass, 0|B\} + \downarrow_{[ON]} = B$

Decompositions

Definition

A coloured vertex is *fixed* if Left can never make it uncoloured.

We can decompose a graph by deleting all fixed vertices and their neighbours.



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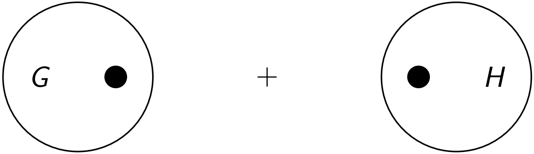

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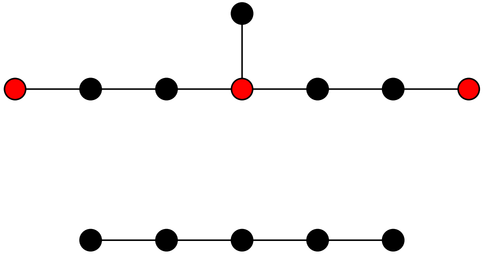
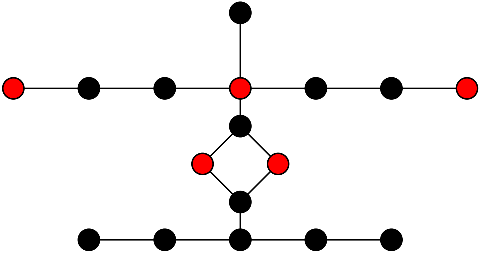


Top component has value 0, bottom component has value under.
Their sum, however, is a Left win.

Decompositions



This is the same game as a connected graph.
So, we must choose between decompositions and disjunctive sums.

A Left-Favourable Construction

Although Right tends to have an advantage for many small graphs, there are numerous ways to construct graphs that are favourable for Left, even if the game begins from an uncoloured graph.

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A is the set of vertices v_i where $i \in \{1, 2, ..., k\}$
There are no edges between vertices in A.
B is the set of vertices $\{x_S | S \subset \{1, 2, ..., k\}\}$
Every possible edge exists between vertices in B.
There is also an edge between v_i and x_S if $i \in S$.

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The idea is for Left to colour $x_\emptyset$. Then Right can only colour the vertices in A. At any point, Left can swap the $x_\emptyset$ with x_T where T is the set of uncoloured vertices of A thereby creating a minimal independent set.

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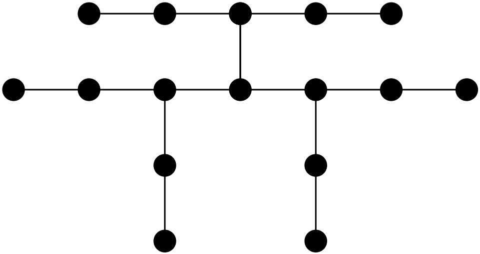
The only exception is if some other component of the graph has value under as there is no sufficiently large k.

A game to try



Open Problems and Generalizations

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- Is it possible to classify all the values that can occur?
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A generalization (for which little is known):

- Right may colour vertices any of k different colours.
- Game ends when the graph has a maximal colouring.

Note that in this variation, Left may not always has a pass move.

Thank you!